Ex. Find the general solution to $y'' + 2y' + 5y = 10t^3 + 2t^2 - t - 1$

Step 1 Solve the associated homogeneous ODE

$$y'' + 2y' + 5y = 0$$

Characteristic equation: $r^2 + 2r + 5 = 0$

$$r = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$$\therefore\ y_h(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$$

Step 2 Find a particular solution. Guess $Y(t) = At^3 + Bt^2 + Ct + D$

$$Y'(t) = 3At^2 + 2Bt + C$$
$$Y''(t) = 6At + 2B$$

$$(6At + 2B) + 2(3At^2 + 2Bt + C) + 5(At^3 + Bt^2 + Ct + D) = 10t^3 + 2t^2 - t - 1$$

$$5At^3 + (6A + 5B)t^2 + (6A + 4B + 5C)t + (2B + 2C + 5D) = 10t^3 + 2t^2 - t - 1$$

$$\begin{cases} 5A = 10 & \Rightarrow A = 2 \\ 6A + 5B = 2 & \Rightarrow 12 + 5B = 2 \Rightarrow B = -2 \\ 6A + 4B + 5C = -1 & \Rightarrow 12 - 8 + 5C = -1 \Rightarrow C = -1 \\ 2B + 2C + 5D = -1 & \Rightarrow -4 - 2 + 5D = -1 \Rightarrow D = 1 \end{cases}$$

$$\therefore\ Y(t) = 2t^3 - 2t^2 - t + 1$$

Step 3 $y(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t + 2t^3 - 2t^2 - t + 1$

Step 4 Use initial conditions to solve for c_1 and c_2. (not applicable to this problem)

① You can see why we guess this particular form — we get an upper triangular system of $n+1$ equations in $n+1$ unknowns.

② Why does this technique only work on ODEs with constant coefficients?

Why are rules Ⓑ and Ⓒ necessary?

Ex What goes wrong if we try $\Psi(t) = At^2 + Bt + C$ as a solution of

$$y'' + 4y' = t^2 \quad ?$$

WRONG ANSWER: $\Psi(t) = At^2 + Bt + C$

$$\Psi'(t) = 2At + B$$

$$\Psi''(t) = 2A$$

$$2A + 4(2At + B) = t^2$$

Step 1

$$y'' + 4y' = 0$$

$$r^2 + 4r = 0$$

$$r(r+4) = 0$$

$$r = 0, -4$$

$$y_h(t) = c_1 e^{0t} + c_2 e^{-4t}$$

$$= c_1 + c_2 e^{-4t}$$

Ex What should the form of the particular solution be?

$$y'' + 4y' = t^2$$

$$\Psi(t) = t(At^2 + Bt + C)$$

$$= At^3 + Bt^2 + Ct$$

$$\Psi' = 3At^2 + 2Bt + C$$

$$\Psi'' = 6At + 2B$$

$$(6At + 2B) + 4(3At^2 + 2Bt + C) = t^2$$

Wrong Answer #2

$\Psi(t) = At^3 + Bt^2 + Ct + D$ ✗

As an exercise, see what goes wrong with this.

$$\begin{cases} 12A = 1 & \Rightarrow A = \frac{1}{12} \\ 6A + 8B = 0 & \Rightarrow B = \frac{1}{2} + 8B = 0 \Rightarrow B = -\frac{1}{16} \\ 2B + 4C = 0 & \Rightarrow -\frac{1}{8} + 4C = 0 \Rightarrow C = \frac{1}{32} \end{cases}$$

$$\therefore \Psi(t) = t\left(\frac{1}{12}t^2 - \frac{1}{16}t + \frac{1}{32}\right) = \frac{1}{12}t^3 - \frac{1}{16}t^2 + \frac{1}{32}t$$

General Solution: $y(t) = c_1 + c_2 e^{-4t} + \frac{1}{12}t^3 - \frac{1}{16}t^2 + \frac{1}{32}t$

Math 3410
Lecture #11
§3.6, 3.7

Case 2 $g(t) = P_n(t)e^{\alpha t}$, where $P_n(t)$ = polynomial of degree n.

Then guess $Y(t) = t^s[A_n t^n + \dots + A_1 t + A_0]e^{\alpha t}$,

where s is the order of α as a root of the characteristic equation.

Ex Find a particular solution to $2y'' + y' - y = 1e^{3t}$

Sol'n Try $Y(t) = \underline{A}\,e^{3t}$ since 3 is not a root of the characteristic equation.

Step 1 $2r^2 + r - 1 = 0$, or $(2r-1)(r+1) = 0$ $r = \frac{1}{2}, -1$

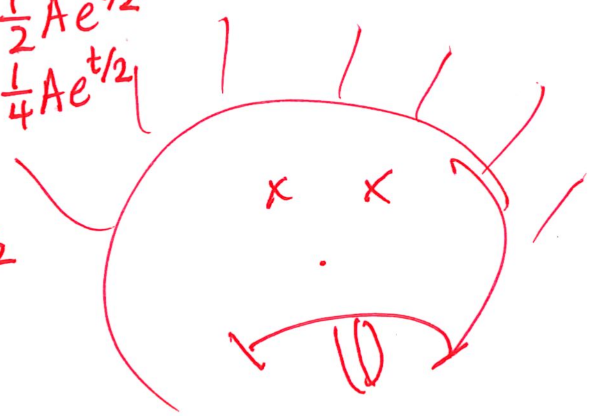
$y_h(t) = c_1 e^{t/2} + c_2 e^{-t}$

Step 2
$$Y(t) = Ae^{3t} \qquad 2Y'' + Y' - Y = e^{3t}$$
$$Y'(t) = 3Ae^{3t} \qquad 18Ae^{3t} + 3Ae^{3t} - Ae^{3t} = e^{3t}$$
$$Y''(t) = 9Ae^{3t} \qquad 20Ae^{3t} = e^{3t} \Rightarrow A = \frac{1}{20}$$
$$\therefore Y(t) = \frac{1}{20}e^{3t}$$

Step 3 $y(t) = c_1 e^{t/2} + c_2 e^{-t} + \frac{1}{20}e^{3t}$

Ex Find a particular solution to $2y'' + y' - y = e^{t/2}$

WRONG ANSWER: Try $Y(t) = Ae^{t/2}$
$$Y'(t) = \frac{1}{2}Ae^{t/2}$$
$$Y''(t) = \frac{1}{4}Ae^{t/2}$$



$$2Y'' + Y' - Y = e^{t/2}$$
$$\frac{1}{2}Ae^{t/2} + \frac{1}{2}Ae^{t/2} - Ae^{t/2} = e^{t/2}$$
$$0 = e^{t/2}$$

What went wrong?

since $r=\frac{1}{2}$ is a root of order 1 of the characteristic equation

<u>Sol'n</u> Try $\Psi(t) = t^1 \cdot Ae^{t/2} = Ate^{t/2}$

$$\Psi'(t) = Ae^{t/2} + At\cdot\frac{1}{2}e^{t/2} = Ae^{t/2} + \frac{1}{2}Ate^{t/2}$$

$$\Psi''(t) = \frac{1}{2}Ae^{t/2} + \frac{1}{2}Ae^{t/2} + \frac{1}{4}Ate^{t/2} = Ae^{t/2} + \frac{1}{4}Ate^{t/2}$$

$$(uv)' = 1u'v + 1uv'$$

(Trick: $(uv)'' = u''v + u'v' + u'v' + uv'' = 1u''v + 2u'v' + 1uv''$)

$$(uv)''' = u'''v + 3u''v' + 3u'v'' + uv'''$$

$$2\Psi'' + \Psi' - \Psi = e^{t/2}$$

$$2Ae^{t/2} + \underline{\frac{1}{2}Ate^{t/2}} + Ae^{t/2} + \underline{\frac{1}{2}Ate^{t/2}} - \underline{Ate^{t/2}} = e^{t/2}$$

$$3Ae^{t/2} = e^{t/2}$$

$$A = \frac{1}{3} \qquad \therefore \boxed{\Psi(t) = \frac{1}{3}te^{t/2}}$$

<u>Ex</u> Find a particular solution to $2y'' + y' - y = -t^2 + e^{3t} + e^{t/2}$

<u>Method #1</u> $\Psi(t) = \underbrace{At^2 + Bt + C}_{t^2+2t+6} + \underbrace{De^{3t}}_{\frac{1}{20}e^{3t}} + \underbrace{Ete^{t/2}}_{\frac{1}{3}te^{t/2}}$

From previous examples $\longrightarrow$

<u>Case 3</u> $g(t) = P_n(t)e^{\alpha t}\cos\beta t$ or $g(t) = P_n(t)e^{\alpha t}\sin\beta t$

Then try $Y(t) = t^s\left[u_1(t)e^{\alpha t}\cos\beta t + u_2(t)e^{\alpha t}\sin\beta t\right]$,

where u_1 and u_2 are polynomials of degree n and s is the order of $\alpha + i\beta$ as a root of the characteristic equation.

<u>Ex</u> Find a particular solution to $y'' + 2y' + 5y = \sin 2t = 1 \cdot e^{0t}\sin 2t$

<u>Step 1</u> $r^2 + 2r + 5 = 0$, $r = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$

<u>NOTE</u> $0 + 2i$ is not a root, so I do not multiply by an extra t^s.

<u>Step 2</u> $Y(t) = Ae^{0t}\cos 2t + Be^{0t}\sin 2t = A\cos 2t + B\sin 2t$

$Y'(t) = -2A\sin 2t + 2B\cos 2t$

$Y''(t) = -4A\cos 2t - 4B\sin 2t$

$Y'' + 2Y' + 5Y = \sin 2t$

$-4A\cos 2t - 4B\sin 2t + 2(-2A\sin 2t + 2B\cos 2t) + 5(A\cos 2t + B\sin 2t) = \sin 2t$

$(-4A + 4B + 5A)\cos 2t + (-4B - 4A + 5B)\sin 2t = \sin 2t$

$$\begin{cases} A + 4B = 0 \\ -4A + B = 1 \end{cases} \qquad \begin{cases} 4A + 16B = 0 \\ -4A + B = 1 \end{cases}$$

$17B = 1$

$B = \frac{1}{17}$ $\qquad A = \frac{-4}{17}$

$\therefore \boxed{Y(t) = -\frac{4}{17}\cos 2t + \frac{1}{17}\sin 2t}$

Questions: Would this technique work if the RHS is $\sin 2t + 4\cos 2t$?

or $-6\sin 2t + 8\cos 2t$?

Ex Find a particular solution to $y'' + y = \sin 2t$

WRONG ANSWER: Try $\Psi(t) = A\cos 2t + B\sin 2t$

CORRECT ANSWER: Try $\Psi(t) =$